

# Dragging of inertial frames in the composed black-hole-ring system

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A well-established phenomenon in general relativity is the dragging of inertial frames by a spinning object. In particular, due to the dragging of inertial frames by a ring orbiting a central black hole, the angular-velocity  $\Omega_{\text{H}}^{\text{BH-ring}}$  of the black-hole horizon in the composed black-hole-ring system is *no* longer related to the black-hole angular-momentum  $J_{\text{H}}$  by the simple Kerr-like (vacuum) relation  $\Omega_{\text{H}}^{\text{Kerr}}(J_{\text{H}}) = J_{\text{H}}/2M^2R_{\text{H}}$  (here  $M$  and  $R_{\text{H}}$  are the mass and horizon-radius of the black hole, respectively). Will has performed a perturbative treatment of the composed black-hole-ring system in the regime of *slowly* rotating black holes and found the explicit relation  $\Omega_{\text{H}}^{\text{BH-ring}}(J_{\text{H}} = 0, J_{\text{R}}, R) = 2J_{\text{R}}/R^3$  for the angular-velocity of a central black hole with zero angular-momentum, where  $J_{\text{R}}$  and  $R$  are respectively the angular-momentum of the orbiting ring and its proper circumferential radius. Analyzing a sequence of black-hole-ring configurations with adiabatically varying (decreasing) circumferential radii, we show that the expression found by Will for  $\Omega_{\text{H}}^{\text{BH-ring}}(J_{\text{H}} = 0, J_{\text{R}}, R)$  implies a *smooth* transition of the central black-hole angular-velocity from its asymptotic near-horizon value  $\Omega_{\text{H}}^{\text{BH-ring}}(J_{\text{H}} = 0, J_{\text{R}}, R \rightarrow R_{\text{H}}^+) \rightarrow 2J_{\text{R}}/R_{\text{H}}^3$  (that is, just *before* the assimilation of the ring by the central black hole), to its final Kerr (vacuum) value  $\Omega_{\text{H}}^{\text{Kerr}}(J_{\text{H}}^{\text{new}}) = J_{\text{H}}^{\text{new}}/2M^{\text{new}2}R_{\text{H}}^{\text{new}}$  [that is, *after* the adiabatic assimilation of the ring by the central black hole. Here  $J_{\text{H}}^{\text{new}} = J_{\text{R}}$ ,  $M^{\text{new}}$ , and  $R_{\text{H}}^{\text{new}}$  are the new parameters of the resulting Kerr (vacuum) black hole after it assimilated the orbiting ring]. We use this important observation in order to generalize the result of Will to the regime of black-hole-ring configurations in which the central black holes possess non-zero angular momenta. In particular, it is shown that the continuity argument (namely, the characteristic *smooth* evolution of the black-hole angular-velocity during an adiabatic assimilation process of the ring into the central black hole) yields a concrete prediction for the angular-velocity/angular-momentum asymptotic functional relation  $\Omega_{\text{H}}^{\text{BH-ring}} = \Omega_{\text{H}}^{\text{BH-ring}}(J_{\text{H}}, J_{\text{R}}, R \rightarrow R_{\text{H}}^+)$  of *generic* (that is, with  $J_{\text{H}} \neq 0$ ) black-hole-ring configurations. Remarkably, we find the simple *universal* relation  $\Delta\Omega_{\text{H}} \equiv \Omega_{\text{H}}^{\text{BH-ring}}(J_{\text{H}}, J_{\text{R}}, R \rightarrow R_{\text{H}}^+) - \Omega_{\text{H}}^{\text{Kerr}}(J_{\text{H}}) = J_{\text{R}}/4M^3$  for the asymptotic deviation of the black-hole angular-velocity in the composed black-hole-ring system from the corresponding angular-velocity of the unperturbed (vacuum) Kerr black hole with the *same* angular-momentum.

## I. INTRODUCTION

The gravitational two-body problem has attracted much attention over the years from both physicists and mathematicians. In particular, it is highly important to explore the physics of a central black hole surrounded by an orbiting ring: it is expected that this composed two-body system may be formed as an intermediate stage in the gravitational collapse of a compact spinning star to form a black hole [1–3]. Likewise, the coalescence of two compact objects may produce a composed black-hole-ring system [1–3]. In addition to these astrophysical motivations, it is highly interesting to explore the composed black-hole-ring system in order to understand how an exterior matter configuration affects the physical properties of central black holes [1–5].

The general-relativistic problem of a *slowly* spinning black hole surrounded by a thin orbiting ring was studied perturbatively by Will [4, 5] (see also [6]). It was shown in [4] that, due to the well-known phenomenon of *dragging* of inertial frames by the orbiting ring, the angular-velocity  $\Omega_{\text{H}}^{\text{BH-ring}}$  [7] of the central black hole in the composed black-hole-ring system is *no* longer related to the black-hole angular-momentum  $J_{\text{H}}$  by the simple

Kerr-like (vacuum) relation

$$\Omega_{\text{H}}^{\text{Kerr}}(J_{\text{H}}) = \frac{J_{\text{H}}}{2M^2R_{\text{H}}} . \quad (1)$$

[Here  $M$  and  $R_{\text{H}} = M + (M^2 - J_{\text{H}}^2/M^2)^{1/2}$  are the mass and horizon-radius of the black hole, respectively]. In particular, Will [4] has demonstrated explicitly that, in the composed black-hole-ring system, one can have a central black hole with zero angular-momentum but with a non-zero angular-velocity [8]

$$\Omega_{\text{H}}^{\text{BH-ring}}(J_{\text{H}} = 0, J_{\text{R}}, R) = \frac{2J_{\text{R}}}{R^3} , \quad (2)$$

where  $J_{\text{R}}$  and  $R$  are respectively the angular-momentum of the orbiting ring and its proper circumferential radius. To the best of our knowledge, no exact (*analytical*) calculations of the frame-dragging effect have been performed for *generic* black-hole-ring configurations (that is, for the case of central black holes with non-negligible angular momenta).

## II. THE CONTINUOUS (SMOOTH) BEHAVIOR OF THE BLACK-HOLE ANGULAR-VELOCITY

The main goal of the present paper is to generalize the result (2) of [4] to the regime of composed black-hole-ring configurations in which the central black holes possess non-zero angular momenta. In particular, we shall use a simple continuity argument in order to provide a concrete analytical prediction for the angular-velocity/angular-momentum asymptotic functional relation  $\Omega_H^{\text{BH-ring}} = \Omega_H^{\text{BH-ring}}(J_H, J_R, R \rightarrow R_H^+)$  of generic (that is, with  $J_H \neq 0$ ) central black holes in the composed black-hole-ring system.

Our approach here is based on a *continuity* argument for the behavior of the black-hole angular-velocity in an *adiabatic* process in which the orbiting ring is assimilated (adiabatically lowered) into the central black hole. In order to demonstrate the idea, we shall first analyze the analytical relation (2) of [4] for the angular-velocity of a zero angular-momentum ( $J_H = 0$ ) central black hole.

Let us first consider a sequence of black-hole-ring configurations with adiabatically varying (decreasing) circumferential radii. Inspection of Eq. (2) reveals that, for a given value of the ring angular-momentum  $J_R$ , the central black-hole angular-velocity *increases* as the ring approaches the black-hole horizon (that is, as  $R$  decreases). In particular, taking the near-horizon limit  $R \rightarrow R_H^+$  in (2), one finds [9]

$$\Omega_H^{\text{BH-ring}}(J_H = 0, J_R, R \rightarrow R_H^+) \rightarrow \frac{J_R}{4M^3} \quad (3)$$

for the angular-velocity of the central black hole just *before* it assimilates the ring.

Let us now calculate the new angular-velocity  $\Omega_H^{\text{Kerr}}(J_H^{\text{new}})$  of the resulting Kerr (vacuum) black hole *after* it absorbed the ring. The adiabatic assimilation of the rotating ring by the central black hole produces the following changes in the black-hole physical parameters:

$$M \rightarrow M^{\text{new}} = M + \mathcal{E}_R \quad \text{and} \quad J_H = 0 \rightarrow J_H^{\text{new}} = J_R, \quad (4)$$

where the energy  $\mathcal{E}_R$  of the rotating ring at the absorption point  $R = R_H$  is given by [10–13]

$$\mathcal{E}_R = \frac{J_H}{2M^2 R_H} \cdot J_R \rightarrow 0 \quad \text{for} \quad J_H = 0. \quad (5)$$

Substituting (4) and (5) into (1), one finds [9]

$$\Omega_H^{\text{Kerr}}(J_H^{\text{new}} = J_R) = \frac{J_R}{4M^3} \quad (6)$$

for the angular-velocity of the final (vacuum) Kerr black hole [14].

Comparing the near-horizon asymptotic ( $R \rightarrow R_H^+$ ) expression (3) for the angular-velocity of the central black hole in the composed black-hole-ring system just *before* the assimilation of the ring, with the expression (6) for the angular-velocity of the resulting Kerr (vacuum) black

hole *after* it assimilated the ring, one concludes that the black hole is characterized by a *smooth* (continuous) evolution of its angular-velocity during the adiabatic assimilation process. That is,

$$\Omega_H^{\text{Kerr}}(J_H^{\text{new}} = J_R) = \Omega_H^{\text{BH-ring}}(J_H = 0, J_R, R \rightarrow R_H^+) . \quad (7)$$

## III. THE ANGULAR-VELOCITY/ANGULAR-MOMENTUM RELATION FOR GENERIC BLACK-HOLE-RING CONFIGURATIONS

In the present section we shall analyze the angular-velocity/angular-momentum asymptotic functional relation  $\Omega_H^{\text{BH-ring}} = \Omega_H^{\text{BH-ring}}(J_H, J_R, R \rightarrow R_H^+)$  of generic (that is, with  $J_H \neq 0$ ) central black holes in the composed black-hole-ring system. To that end, we shall use the characteristic *continuity* relation [15]

$$\Omega_H^{\text{Kerr}}(J_H^{\text{new}} = J_H + J_R) = \Omega_H^{\text{BH-ring}}(J_H, J_R, R \rightarrow R_H^+) \quad (8)$$

for the evolution of the black-hole angular-velocity during an *adiabatic* assimilation process of the orbiting ring by the central black hole.

We consider a composed black-hole-ring system which is characterized by the physical parameters  $J_H, J_R$ , and  $R$ . The adiabatic absorption of the ring by the central black hole produces a final Kerr (vacuum) black hole with the following parameters:

$$M \rightarrow M^{\text{new}} = M + \mathcal{E}_R \quad \text{and} \quad J_H \rightarrow J_H^{\text{new}} = J_H + J_R, \quad (9)$$

where the energy  $\mathcal{E}_R$  of the ring at the absorption point  $R = R_H$  is given by [10–13]

$$\mathcal{E}_R = \frac{J_H}{2M^2 R_H} \cdot J_R. \quad (10)$$

Substituting (9) and (10) into (1), one finds

$$\Omega_H^{\text{Kerr}}(J_H^{\text{new}} = J_H + J_R) = \frac{J_H}{2M^2 R_H} + \frac{J_R}{4M^3} \quad (11)$$

for the angular-velocity of the final (vacuum) Kerr black hole [14].

Taking cognizance of Eqs. (1) and (11), and using the *continuity* argument (8) for the evolution of the black-hole angular-velocity during the *adiabatic* assimilation process of the ring into the central black hole, one finds the characteristic angular-velocity/angular-momentum asymptotic relation

$$\Omega_H^{\text{BH-ring}}(J_H, J_R, R \rightarrow R_H^+) = \Omega_H^{\text{Kerr}}(J_H) + \frac{J_R}{4M^3} \quad (12)$$

for a central black hole of angular-momentum  $J_H$  in the composed black-hole-ring system [Here  $\Omega_H^{\text{Kerr}}(J_H)$ , as given by (1), is the angular-velocity of a (*vacuum*) Kerr black hole with the *same* angular-momentum].

#### IV. SUMMARY AND DISCUSSION

The composed black-hole-ring system is one of the most fundamental problems in general relativity and astrophysics [1–3]. This two-body system is characterized by one of the most intriguing phenomena in general relativity, namely the *dragging* of inertial frames. In a very interesting work, Will [4, 5] studied this composed system perturbatively in the regime of *slowly* spinning central black holes. It was shown in [4, 5] that the effect of dragging of inertial frames by the orbiting ring yields a non-trivial angular-velocity/angular-momentum relation for the central black hole. In particular, Will [4, 5] found the non-zero angular-velocity (2) for a central black hole of zero angular-momentum ( $J_H = 0$ ) in the composed black-hole-ring system.

To the best of our knowledge, in the physical literature there are no available analytical results for the frame-dragging effect in *generic* black-hole-ring configurations (that is, for central black holes with non-negligible angular momenta). The main goal of the present paper was to generalize the result (2) of Will [4, 5] to the regime of composed black-hole-ring configurations in which the central black holes possess non-zero angular momenta.

In particular, we have explored the angular-velocity/angular-momentum asymptotic functional relation  $\Omega_H^{\text{BH-ring}} = \Omega_H^{\text{BH-ring}}(J_H, J_R, R \rightarrow R_H^+)$  of generic black-hole-ring configurations. To that end, we have used a *continuity* argument [16] for the evolution of the black-hole angular-velocity during a physical process in which the orbiting ring is adiabatically lowered into the central black hole. This continuity argument [see Eqs. (7) and (8)] yields the non-trivial (non Kerr-like) angular-velocity/angular-momentum asymptotic functional relation (12) for *generic* (that is, with  $J_H \neq 0$ ) central black holes in the composed black-hole-ring system.

Remarkably, our result (12) for the angular velocity

of the perturbed central black hole implies the simple *universal* [17] relation

$$\Delta\Omega_H(R \rightarrow R_H^+) = \frac{J_R}{4M^3}, \quad (13)$$

where  $\Delta\Omega_H(R \rightarrow R_H^+) \equiv \Omega_H^{\text{BH-ring}}(J_H, J_R, R \rightarrow R_H^+) - \Omega_H^{\text{Kerr}}(J_H)$  is the asymptotic deviation of the black-hole angular-velocity  $\Omega_H^{\text{BH-ring}}(J_H)$  in the composed black-hole-ring system from the corresponding angular-velocity  $\Omega_H^{\text{Kerr}}(J_H)$  [see Eq. (1)] of the unperturbed (vacuum) Kerr black hole with the same angular-momentum  $J_H$ . It is worth emphasizing that the asymptotic relation (13) for  $\Delta\Omega_H$  is *universal* in the sense that it is *independent* of the black-hole angular-momentum  $J_H$ .

Finally, we would like to end this paper with a *conjecture*. In particular, we would like to suggest a simple (and compact) formula which generalizes the asymptotic near-horizon result (13) to generic values of the ring radius  $R$ . To that end, we note that the simplest [18] functional relation  $\Delta\Omega_H = \Delta\Omega_H(R)$  which reduces to (2) in the zero angular-momentum  $J_H \rightarrow 0$  limit [19], and to (13) in the asymptotic near-horizon  $R \rightarrow R_H$  limit, is given by [20]

$$\Delta\Omega_H(R) = \frac{J_R}{4M^3} \cdot \left(\frac{R_H}{R}\right)^3. \quad (14)$$

It would be highly interesting to test the validity of our conjectured relation (14) with full non-linear [21] numerical computations for generic (that is, with  $J_H \neq 0$ ) black-hole-ring configurations.

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  - [7] We use the symbol  $\Omega_H^{\text{BH-ring}}$  to denote the angular-velocity of the black-hole horizon in the composed black-hole-ring system.
  - [8] This results is valid in the perturbative regime  $J_R/R^2 \ll 1$ . There are also subleading correction terms of order  $O(J_R^2/R^5)$  on the r.h.s of this relation.
  - [9] Here we have used the leading-order relation  $R_H = 2M$  for the horizon-radius of the zero angular-momentum black hole.
  - [10] B. Carter, Phys. Rev. **174**, 1559 (1968).
  - [11] We consider here an adiabatic assimilation process of the ring into the central black hole. In this idealized scenario the assimilated ring has a *zero* radial momentum at the point of capture.
  - [12] It is worth emphasizing that the mass-energy of the ring as measured by asymptotic observers,  $\sqrt{g_{00}}\mu$ , is completely red-shifted ( $\sqrt{g_{00}} = 0$ ) at the absorption point (at the black-hole horizon).
  - [13] This results is valid in the perturbative regime  $J_R/R^2 \ll 1$ . There are also small (sub-leading) correction terms of order  $O(J_R^2/R^3, \mu^2/R)$  on the r.h.s of this relation.
  - [14] Note that the black-hole surface area  $8\pi MR_H$  plays the role of an adiabatic invariant [see, J. D. Bekenstein, “Black Holes: Classical Properties, Thermodynamics and Heuristic Quantization”, in *Cosmology and Gravitation*, M. Novello, ed. (Atlantis Sciences, France 2000),

- pp. 1-85 (gr-qc/9808028)]. It therefore acquires small (sub-leading)  $O(J_R^2/M^2, \mu^2)$  corrections, which are only second-order in the ring parameters.
- [15] Note that the r.h.s of (8) refers to the angular-velocity of the central black hole in the composed black-hole-ring system in the near-horizon limit  $R \rightarrow R_H^+$  (that is, just *before* the assimilation of the ring by the black hole), whereas the l.h.s of (8) refers to the angular-velocity of the final Kerr (vacuum) black hole (that is, *after* it assimilated the ring).
  - [16] That is, a *smooth* functional behavior [see Eqs. (7) and (8)] of the central black-hole angular-velocity in an adiabatic assimilation process of the orbiting ring into the central black hole.
  - [17] Note that the relation (13) for  $\Delta\Omega_H$  is universal in the sense that it is *independent* of the black-hole angular-momentum  $J_H$ .
  - [18] It should be emphasized that the conjectured relation (14) is not unique. For example, the more complicated expression  $\Delta\Omega_H(R) = \frac{RR_H J_R}{M(R^2 + J_H^2/M^2)^2}$  also reduces to (2) in the zero angular-momentum  $J_H \rightarrow 0$  limit (which corresponds to  $R_H \rightarrow 2M$ ), and to (13) in the asymptotic near-horizon  $R \rightarrow R_H$  limit.
  - [19] Note that this limit corresponds to  $R_H \rightarrow 2M$ .
  - [20] Note that  $R_H \rightarrow 2M$  in the zero angular-momentum  $J_H \rightarrow 0$  limit, in which case (14) reduces to (2). Likewise,  $R \rightarrow R_H$  in the near-horizon limit, in which case (14) reduces to (13).
  - [21] It is worth emphasizing again that our analysis is valid in the perturbative regime  $J_R/R^2 \ll 1$ .